

## Superluminal travel made possible (in two dimensions)

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**ABSTRACT:** We argue that superluminal signal propagation is possible in consistent Poincare invariant quantum field theories in two space-time dimensions, provided spatial parity is broken. This happens due to existence of the “instantaneous” causal structure, with one of the light cone variables being a global time. In two dimensions this causal structure is invariant under the Poincare group if one gives up the spatial parity. As a non-trivial example of a consistent interacting quantum field theory with this causal structure we discuss a non-linear  $SO(1, 1)$  sigma-model, where  $SO(1, 1)$  is the Lorentz symmetry. We show that this theory is asymptotically free and argue that this model is also well defined non-perturbatively, at least for some values of parameters. It provides an example of a microscopic Poincare invariant quantum field theory with local action, but non-local physical properties. Being coupled to gravity this “instantaneous” theory mixes with the Liouville field. If proves to be consistent, the resulting model can be used to construct (non-critical) string theories with very unconventional properties by introducing the instantaneous causal structure on the world-sheet.

**KEYWORDS:** Field Theories in Lower Dimensions, Space-Time Symmetries, Sigma Models, 2D Gravity.

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## 1. Introduction

The question we will address in the current paper is

*Whether a superluminal signal propagation is possible in a consistent Poincare invariant quantum field theory?*

This question may appear rather provocative as it seems to be known that the answer is *No*. Also the physics to which this question will lead us turns out to be quite unconventional, so let us start with explaining our motivations. The very first motivation (or, rather, an excuse) for asking this question is that we claim that the answer is *Yes*: superluminal, and even instantaneous signal propagation is possible, at least in two space-time dimensions with broken spatial parity.

Apart from this rather unexpected answer there is a number of other motivations more directly linked to the known physics. First, there are reasons to suspect that locality is

only an approximate notion in gravitational theories. Probably the strongest arguments supporting this viewpoint are the absence of sharply defined local observables in theories with dynamical gravity, and the indication that the information recovery during a black hole evaporation requires a certain degree of non-locality (see e.g., [1, 2] for recent discussions). On the other hand, in a Poincare invariant theory non-locality in space would imply also non-locality in time that appears to be at odds with unitary time evolution and causality. To analyze this problem it is useful to construct explicit models with non-local properties. In this paper we will present explicit models exhibiting instantaneous signal propagation, that definitely qualifies as a non-local effect. To the best of our knowledge, these models provide the first example of Poincare invariant microscopic quantum field theories with a truly non-local behavior, if as yet in two dimensions.

A more general motivation to ask a question like that is a theoretical curiosity triggered by the following considerations. In the early years of GUT and string theory there was a hope that the consistency conditions for coexistence of gravity and quantum field theory uniquely fix the Standard Model as the only possible low energy effective field theory in a world with gravity. However, later developments such as D-branes [3] revealed that the power of mathematical consistency was overestimated and a number of low energy field theories that can be embedded in a microscopic gravitational theory is much larger. This trend has apparently reached its top with the discovery of a vast landscape of metastable de Sitter vacua [4].

At this point it is natural to ask where is the boundary (if any) of the set of consistent gravitational theories. A plausible answer to this question may be the “swampland” idea [5]. Namely, it well may be that instead of directly predicting the spectrum and/or coupling constants of the Standard Model, self-consistency considerations imply a set of relations between them, invisible at the level of an effective field theory. Effective field theories where these relations do not hold would belong to the “swampland” and cannot be UV completed to a theory of quantum gravity. Along these lines several examples of surprising limitations on effective field theories were found [6–8].

Still, an adventurous person may challenge this point of view and suggest that deep inside the swampland one may discover new islands of theories, that are totally different from all what we know at the moment (“wonderland”). In other words one may wonder whether there exist UV complete theories where the principles of quantum field theory we usually take for granted are violated? Clearly, the impossibility of a superluminal signal propagation is one of such principles.

In fact, a number of effective field theories were constructed recently that will either turn out to provide quite interesting examples of the swampland inhabitants, or may be a hint that gravitational theories with rather exotic properties exist. These theories resulted from attempts to find a consistent modification of gravity at long distances, to large extent motivated by the cosmological constant problem. They include the brane world DGP model [9] as well as four dimensional models describing gravity in the Higgs phase, such as the ghost condensate [10] and more general models of massive gravity [11] (see [12] for a review). A closely related class of models is the Einstein-aether theory [13] and its generalizations.

These models appear consistent from the effective field theory viewpoint and provide a number of interesting phenomenological signatures (such as the anomalous precession of the Moon perihelion in the DGP model [14, 15] or a strong monochromatic gravitational wave signal due to primordial massive gravitons [16]). Also they provide a number of interesting theoretical possibilities, for instance, opening a room for an alternative to inflation [17]. On the other hand, it is not clear at all whether these models can be UV completed in a consistent microscopic quantum theory. In fact, there are good reasons [18, 19, 22], based on considerations related to locality, causality and black hole physics, why such a UV completion is extremely hard to achieve. To find a way around these problems (or to conclusively demonstrate that such a way doesn't exist) requires to further scrutinize the standard lore about causality in Lorentz invariant quantum field theory.

Clearly, it is straightforward to construct consistent quantum theories with faster-than-light or even instantaneous signal propagation if one gives up Lorentz invariance. Most non-relativistic quantum mechanical systems have this property. The real challenge is to incorporate superluminal propagation in a theory with gravity. To stress the importance of existence of the universal causal cone in gravitational theory, it suffices to note that if superluminal propagation were allowed, information could be extracted from black holes already at the classical level [19–21] and a remarkable success of black hole thermodynamics in general relativity and string theory would appear to be an unnecessary coincidence. Indeed, the conventional black hole thermodynamics breaks down if there is no causal cone, universal for all fields [19, 22].

If one starts with a gravitational theory and decouples gravity by sending  $M_{Pl}$  to infinity, one ends up with a Poincare invariant quantum field theory. That's why we will focus on Poincare invariant field theories and study whether superluminal propagation is compatible with the Poincare invariance. Well-known arguments reviewed in section 2 suggest that it is impossible. However, there is a couple of loopholes in these arguments leaving a chance for superluminal travel.

The first loophole is that in two space-time dimensions, if one gives up the spatial parity, there are two different causal structures compatible with the remaining (connected) part of the Poincare group. The first is the standard one, and the second is the “instantaneous” causal ordering, such that for a given point the half-plane with a larger value of the global light cone time  $x^+$  is in the causal future, and the half-plane with a smaller value of  $x^+$  is in the causal past. We will refer to this structure as  $x^+$ -ordered causal structure. A natural question arises whether consistent quantum field theories with the second choice of the causal structure exist. We will answer affirmatively to this question and hope to convince the reader that these theories may be quite interesting. This is the main result of the present paper.

Another loophole arises due to the possibility of spontaneous breaking of Lorentz invariance. Models describing gravity in the Higgs phase realize exactly this idea. As we said, in spite of a significant progress achieved recently in constructing such models at the effective field theory level, none of them has been embedded into a UV complete microscopic theory so far. It appears easier to achieve this goal in two space-time dimensions due to a very simple reason: many field theories that are non-renormalizable and require UV

completion in higher dimensions can be renormalized at  $D = 2$ . One may think that this hope is doomed due to the Coleman-Mermin-Wagner (CMW) theorem [24, 25] establishing that spontaneous breaking of a continuous symmetry is not possible in two dimensions. However, as we argue below, in spite of the restoration of the Lorentz invariance at long distances, the short distance physics captures all relevant and interesting properties following from the spontaneous symmetry breaking.

Thus two somewhat different reasons — special geometrical properties and better chances to construct a UV complete theory with spontaneous Lorentz breaking — lead us to study a possibility of superluminal travel in two space-time dimensions. We start by discussing in section 2 how the standard arguments against superluminal signal propagation are avoided in 2d. In section 3 we introduce the model which we consider in the rest of the paper. This is the so-called Einstein-aether theory, a model describing a dynamical vector field that develops a Lorentz-violating vev. This model has been recently studied in two dimensions at the classical level in [26], while our main interest here is whether it can be promoted to a consistent quantum theory. We identify four different regimes of this model corresponding to different choices of parameters. Two choices explicitly break the spatial parity and naturally enjoy the causal structure based on time-ordering with respect to the light-cone coordinate  $x^+$ , while in the third and the fourth cases the action is parity symmetric.

We show in section 4 that at the perturbative level all four parameter choices give rise to renormalizable asymptotically free theories. However, beyond the perturbation theory the four cases are quite different. We consider them separately in section 5.

We start with the case where violation of parity is, in a sense, maximal. This turns out to be the cleanest example of a consistent theory with the instantaneous causal structure. This theory allows a (non-standard) Wick rotation resulting in a path integral with a positive definite real part of the Euclidean action. Under the renormalization group (RG) it flows from a free theory with the standard causal structure in the UV to a weakly coupled instantaneous theory in the IR. Physically, this implies that an observer in such a theory by performing short scale measurements can be tricked into thinking that she lives in a world with the standard causal structure. However, after some time (or increasing the precision of the measurements) she will find out apparent causal paradoxes that are removed after realization what the actual causal structure is. Given that the true causal structure is instantaneous, this model provides an example of a genuinely non-local Poincare invariant quantum field theory.

Then we turn to the other class of parity breaking Einstein-aether models. In the minimal version they exhibit the conventional causal structure, but by introducing an additional field can be easily turned into theories obeying the  $x^+$ -ordered causal structure. The interesting property of these theories is that at the classical level they are invariant under the right moving part of the Virasoro algebra and under the global part of the left mover's Virasoro algebra. At the quantum level, in spite of being derivatively coupled, they can be renormalized by normal ordering. This gives rise to a hope that one may find solvable models with the instantaneous causal structure.

For completeness, we discuss also the parity preserving cases, though they do not

provide theories obeying the  $x^+$ -ordered causal structure. For one choice of parameters the causal structure is the standard one. In IR the theory flows to strong coupling which suggests that a mass gap is generated. Perturbatively, the theory is stable. However, we find that the classical Hamiltonian is unbounded from below indicating a non-perturbative instability of the semiclassical vacuum. This may or may not imply inconsistency of the model for this parameter choice.

Finally, we consider the second parity preserving choice of parameters that leads to a theory where a fixed causal structure is absent altogether. This case is the most subtle of all. As we discuss, it has much in common with the gravitational theories. In particular, the causal structure itself is dynamical: it is determined by a background under consideration. Though the action in this case is parity symmetric, at the quantum level the theory can be consistent if the spatial parity is spontaneously broken. This may be regarded as the two-dimensional analog of a spontaneous breaking of Lorentz symmetry in higher dimensions.

In section 6 we outline what happens when the above models are coupled to gravity. We show that in the conformal gauge the Einstein-aether sector produces a contribution to the Liouville action of the dilaton equivalent to that of a single two-dimensional scalar boson. Besides, the Einstein-aether mode mixes with the Liouville mode. It remains to be understood whether the resulting model allows for a consistent quantization.

We conclude in section 7 by summarizing the lessons obtained from our study, and point out what we think are the promising directions for future work.

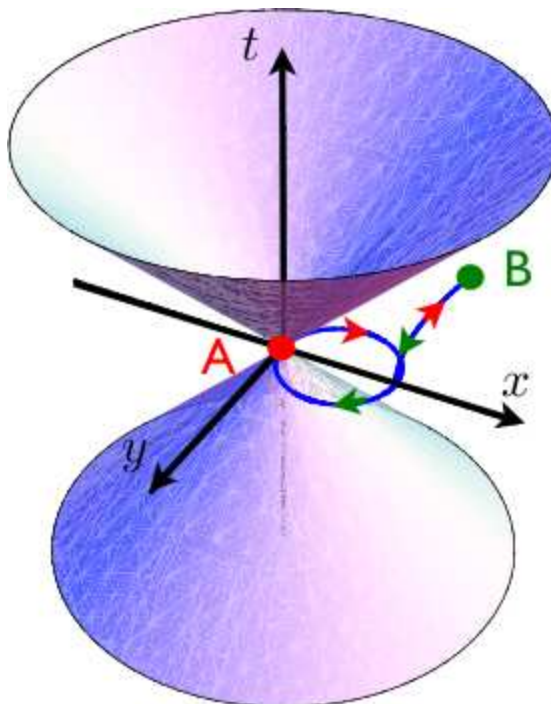
## 2. Superluminality in two dimensions

### 2.1 Why superluminal travel is hard in general

Let us start with a brief review of well-known arguments why superluminal travel is in general problematic in Poincare-invariant theories (the bulk of this discussion follows [18]). Let us first assume that the Lorentz invariance is unbroken and it is possible to transfer a signal from a space-time point A to another point B at the space-like separation. This possibility implies that the point A is in the causal past of the point B. However, there always exists a Lorentz transformation that exchanges points A and B with each other (see figure 1) so that one concludes that the point B should also be in the causal past of the point A. This clearly does not make sense or, formally, in this case it is impossible to introduce a causal ordering of events. This proves that superluminal travel is incompatible with unbroken Lorentz invariance.

One proposal to get around this problem would be to say that superluminal travel is not possible in the Lorentz-invariant vacuum, but excitations in non-trivial backgrounds can nevertheless propagate outside the Minkowski light-cone. A simple example of a classical Lorentz-invariant theory with this type of behavior is provided by the following action describing a single scalar field  $\phi$ ,

$$S = \int d^D x \Lambda^4 P(X), \quad (2.1)$$



**Figure 1:** Any two spacelike separated points A and B can be interchanged by a combination of a Lorentz boost and a spatial rotation. Consequently, no Poincare invariant causal ordering exists such that A and B are causally connected.

where<sup>1</sup>  $X = (\partial_\mu \phi)^2$ . At the quantum level for non-trivial functions  $P$  the theory (2.1) is non-renormalizable and should be considered as an effective low energy theory with a cutoff  $\leq \Lambda$ . In this model scalar perturbations  $\pi$  propagate with the speed of light in the Lorentz-invariant vacuum  $\phi = 0$ ; in a non-trivial background their propagation is described by the wave equation,

$$\square_G \pi = 0, \tag{2.2}$$

where  $\square_G$  is the d’Alambertian with respect to the effective “acoustic” metric (cf. [23])

$$G_{\mu\nu} = \Omega \left( \eta^{\mu\nu} + \frac{2P''}{P'} \partial^\mu \phi \partial^\nu \phi \right)^{-1} = \Omega \left( \eta_{\mu\nu} - \frac{2P'' \partial_\mu \phi \partial_\nu \phi}{P' + 2P'' X} \right), \tag{2.3}$$

with the conformal factor given by

$$\Omega = (P' (P' + 2P'' X))^{1/D-2}.$$

In order to avoid unacceptable ghost or gradient instabilities one needs

$$P' > 0, \quad P' + 2P'' X > 0. \tag{2.4}$$

However, there is no obvious pathology if at some value  $X_0$  the second derivative of  $P$  turns negative,  $P''(X_0) < 0$ . In the corresponding backgrounds (for instance, one can take

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<sup>1</sup>We use the metric signature  $(+, -, -, \dots)$ .

$\phi = X_0^{1/2}t$ ) the causal cone of the effective metric (2.3) is broader than the Minkowski light cone and scalar perturbations can propagate faster than light.

Note that this kind of field theories have a peculiar property that the space-time events don't possess any fixed causal ordering independent of the field configuration. This property is definitely very unusual in the field theory context, however, we are used to this situation in gravitational theories. In particular, at the classical level the absence of the fixed causal structure doesn't prevent one from formulating an initial value problem, provided one uses surfaces that are space-like with respect to the acoustic metric (2.3) as a set of Cauchy slices (see [23] for a recent discussion).

A problem with this proposal is that while it appears viable at the level of effective field theory, it is very difficult to imagine a UV completion of the corresponding effective field theories. Clearly, effective field theories of this kind cannot arise as the low energy limit of conventional Poincare-invariant renormalizable field theories, where microcausality (and, as a consequence, subluminal propagation in all backgrounds) is built in by construction. Moreover, if arbitrary small deformation of the Lorentz invariant vacuum is enough to reach a background with superluminal propagation (this is the case if  $P''(0) < 0$ ), then the  $2 \rightarrow 2$  scattering amplitude in the model (2.1) does not enjoy the conventional analytical properties, that hold not only in any renormalizable field theory, but in the string theory (at least at weak coupling) as well [18].

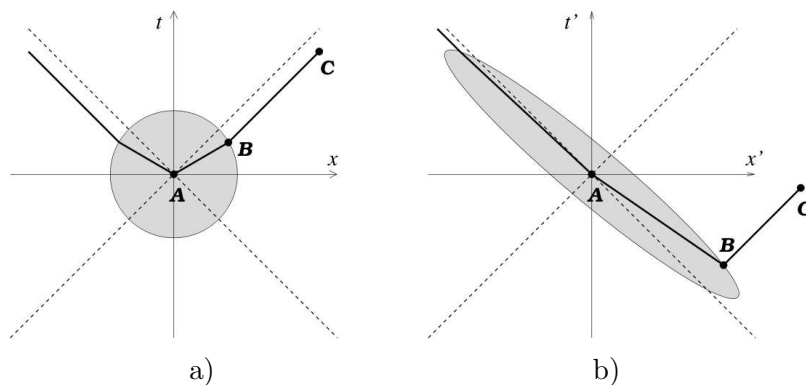
There is yet another problem with superluminal travel in this model. Namely, it is possible to find configurations of the scalar field within the regime of validity of the effective field theory, such that the effective metric (2.3) exhibits closed time-like curves, so that the initial value problem ceases to be globally well-defined. The idea of this construction is similar to the idea of the paradox demonstrating impossibility of superluminal travel in the Lorentz invariant vacuum. One starts with creating a finite region of space where the causal cone is broader than the Minkowski light cone (figure 2a). If one now performs a large enough boost of this field configuration, scalar perturbations will propagate back with respect to the Minkowski time inside the superluminal region (figure 2b). This is not yet a problem by itself, a non-trivial scalar field configuration spontaneously breaks Lorentz invariance and the causal structure induced by the effective metric (2.3) should be used to define the causal ordering in such a background. In more physical terms, one should use the fastest available particles — the scalar quanta in this case — to perform the clock synchronization.

However, if one takes now a pair of such bubbles and boosts them in the opposite directions, one obtains a field configuration with a closed time-like curve (see figure 3). At large enough transverse separation between bubbles all local Lorentz invariant quantities are small, so that this field configuration is within the regime of validity of the effective field theory. One may object that this configuration requires external sources to become a solution of the field equations. However, there appears to be no natural local constraint on the sources that would forbid causal pathologies<sup>2</sup>. These arguments indicate that UV

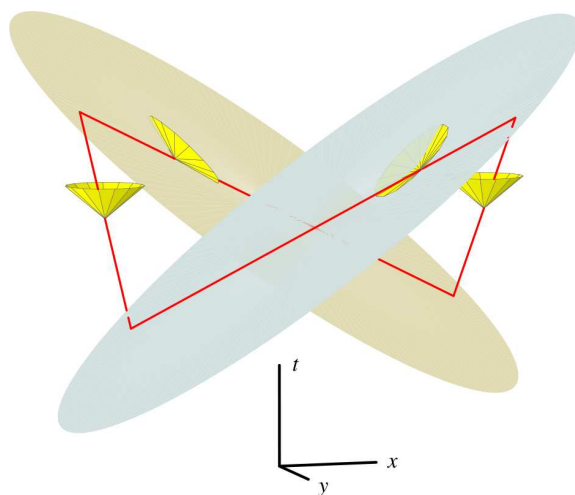
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<sup>2</sup>This is different from the case of general relativity where the null energy condition is a natural candidate to prevent a formation of closed time-like curves starting from the configurations approaching flat space-time





**Figure 2:** a) A non-trivial field configuration gives rise to a finite space-time region with superluminal propagation. b) A large boost of such a configuration results in the signal propagation backward in time.



**Figure 3:** Two regions with superluminal propagation give rise to a closed time-like curve when boosted towards each other.

completion of superluminal effective field theories, if any, should be very different from known quantum field theories or string theory.

One logical possibility to get around both problems, would be to consider a theory where the Lorentz invariant state with  $X = 0$  is absent whatsoever. Problems related to the analytical properties of the scattering amplitudes disappear simply because the Lorentz invariant S-matrix does not exist any longer. Also the above construction of the closed time-like curve heavily relies on the coexistence of Lorentz invariant and Lorentz breaking phases. For instance, if due to some reasons the whole region where  $X \leq 0$  is never accessible, so that  $\partial_\mu \phi$  is necessarily time-like (both with respect to the Minkowski and to the acoustic metric), one can always choose  $\phi$  itself as a global time, so that no causal paradoxes arises. Clearly, none of the known UV complete quantum theories exhibit this behavior.

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in the infinite past [27].

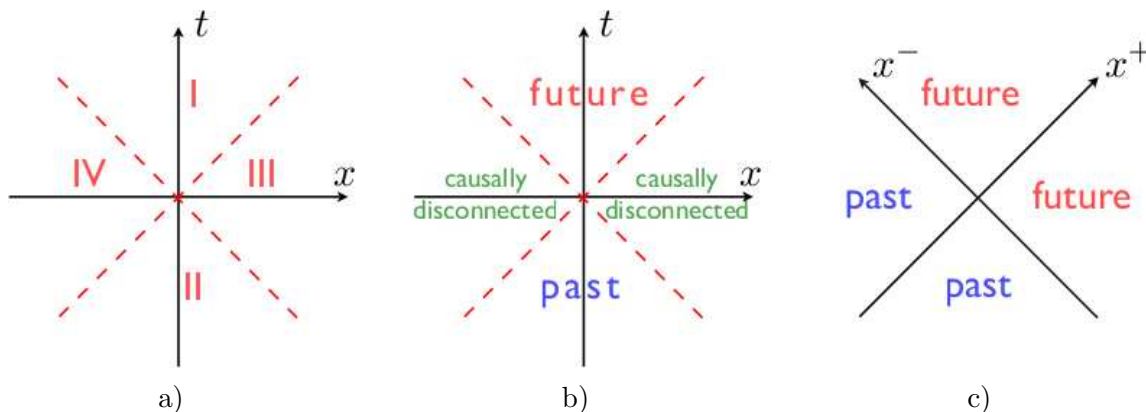
One of the motivations to consider two-dimensional models, is that in two dimensions there are better chances to construct renormalizable theories with this kind of behavior. Indeed, in section 3 we will present a model (of a vector field, rather than of a scalar) that at the classical level does not possess a Lorentz-invariant vacuum state. A well-known feature of two-dimensional physics is that continuous symmetries that appear to be broken spontaneously at the classical level are restored in the quantum theory at long distances. Still, even in the presence of the Poincare-invariant vacuum one can get around the above arguments against superluminality due to peculiar properties of two-dimensional space-time.

## 2.2 Life is special in two dimensions

First, let us see why the argument about closed time-like curves is not applicable in two dimensions. Let us again consider superluminal effective theories of the form (2.1). For simplicity, let us concentrate on *strictly* superluminal theories, i.e. such that  $P'' \leq 0$  for all values of  $X$ . If conditions (2.4), ensuring that an effective field theory is free of local pathologies, are satisfied (this is always the case in the vicinity of the Lorentz-invariant vacuum  $X = 0$ , if this vacuum itself is free of ghosts,  $P'(0) > 0$ ), then locally the metric (2.3) defines a regular causal structure. The vector  $t^\mu = (1, 0)$  is time-like at each point with respect to the effective metric, and points inside the future light cone. Assume now that there exists a closed curve which is everywhere time-like and future-directed with respect to the metric (2.3). Without loss of generality one can assume that this curve has no self-intersections, as all self-intersections can be smoothly deformed out while keeping the curve time-like. But for any closed oriented curve without self-intersections on a plane one can find at least one point where its tangent vector is collinear to the past time-like vector  $-t^\mu = (-1, 0)$ . Thus we arrive at the contradiction which means that it is impossible to construct a closed time-like curve. In more physical terms closed time-like curves are absent because for any setup of the type shown in figure 3 the two bubbles will necessarily collide as in 2d there is no possibility to separate them in the transverse direction. As a result the effective field theory will break down.

This argument suggests that superluminal travelers may have better chances to succeed in two space-time dimensions. What is even more encouraging is that, as we are going to see, in two dimensions superluminal travel is possible even in the state with *unbroken* Lorentz invariance, provided the space parity is broken. This will be the key observation exploited in the current paper.

Indeed, in two dimensions light cones emanating from a given point divide the space-time into *four*, instead of three in higher dimensions, separate regions (see figure 4): interiors of the future and past light cones (regions I and II) and two space-like regions (III and IV). The standard causal structure assumes that the region I is the causal future of the origin  $O$ , the region II is the causal past, while regions III and IV are causally disconnected from  $O$ . Clearly, this choice is compatible with the extended Poincare group including also the spatial parity transformation  $x \rightarrow -x$ . The latter is necessary to map points from the region III into the region IV. Consequently, if the spatial parity is broken, there is no reason for these two regions to be in the same causal relation with  $O$ . Interestingly, there



**Figure 4:** a) Light cones emanating from a given point divide the two-dimensional space-time into four separate regions. As a result two causal orderings are possible: b) the standard and c) the  $x^+$ -ordered causal structure.

is another possible choice of a causal structure, which is fully compatible with a connected part of the Poincare group. Namely, one may choose the union of regions I and III to form the causal future of the origin and the union of regions II and IV to form the causal past. There is no causally disconnected region in this case, so that this choice of causal structure corresponds to the presence of instantaneous interactions.

A peculiar property of this causal structure is that it uniquely fixes the way the space-time is foliated by Cauchy surfaces. These are the surfaces of a constant light-cone variable

$$x^+ = \frac{1}{\sqrt{2}}(t + x) .$$

Consequently this variable plays the role of a global time, and it is natural to call the causal structure introduced above the “ $x^+$ -ordered” causal structure. The second light cone variable

$$x^- = \frac{1}{\sqrt{2}}(t - x)$$

is a space coordinate. It is amusing that in two dimensions the existence of a global time is compatible with Lorentz invariance.

Thus one concludes that in two space-time dimensions superluminal travel is possible even in Lorentz-invariant theories with a fixed causal structure. The study of this possibility is our main goal in the current paper.

A rather trivial example of a field theory with the  $x^+$ -ordered causal structure is

$$S = - \int d^2x (\partial_- A^-)^2 . \tag{2.5}$$

This action is Poincare invariant with  $A^-$  transforming as a  $(-)$  component of a vector. When quantized in the light-cone coordinates, using  $x^+$  as the time variable, the field  $A^-$  mediates an instantaneous linear potential. Of course this model is rather trivial as it does not describe any propagating degrees of freedom. To bring more life into this model we need to couple  $A^-$  to a truly dynamical degree of freedom having a non-trivial  $x^+$  evolution.

One way to achieve this is to write an action of the form

$$S = \int d^2x [-(\partial_- A^-)^2 + (\partial_+ A^+)^2 - m^2(A^- A^+)^2] . \quad (2.6)$$

To gain some insight into the physics of this system note that this action does not involve time ( $= x^+$ ) derivatives of  $A^-$ , so this field can be integrated out. The integral over  $A^-$  is Gaussian, so this procedure reduces to calculating a determinant and results in a highly non-local (in “space” coordinate  $x^-$ ) interaction. It would be interesting to study whether (2.6) or its generalizations provide examples of consistent interacting theories. One may hope that this is the case, given that the Wick rotation  $x^+ \rightarrow -ix^+$  transforms (2.6) into the positive definite Euclidean action. Note that a mass term  $\mu^2 A^- A^+$  would spoil this property and introduce instabilities. However, such a term may be forbidden by a discrete symmetry  $A^- \mapsto -A^-$ .

Note also, that the conventional Wick rotation  $t \rightarrow -it$  would not give rise to a positive definite Euclidean action. More generally, a necessary prerequisite for the superluminal propagation is that it should not be possible to perform a conventional Wick rotation resulting in a real positive definite Euclidean action. Indeed, superluminal propagation means that a retarded propagator of a certain field  $\phi$  does not vanish outside the light cone. Equivalently, the  $T$ -ordered two point function of  $\phi$  has an imaginary part outside the light cone. If one could make a conventional Wick rotation resulting in the positive real action, it would be possible to calculate the  $T$ -ordered product outside the light cone directly in the Euclidean space with a manifestly real result [28].

In spite of the spatial parity breaking, there is no problem to rewrite the action (2.6) in the covariant form by making use of the antisymmetric Levi-Civita tensor  $\epsilon^{\mu\nu}$ , and introduce coupling to gravity,

$$S = \int d^2x \sqrt{-g} \left\{ -\nabla_\mu A^\mu \frac{\epsilon^{\nu\lambda}}{\sqrt{-g}} \nabla_\nu A_\lambda - \frac{m^2}{4} (g_{\mu\nu} A^\mu A^\nu)^2 \right\} \quad (2.7)$$

In the current paper we proceed to a somewhat more involved set of models exhibiting the same causal structure. One reason is that, as we hope to convince the reader, these models are interesting on their own right. A more concrete motivation is that we would like to use superluminal theories as a toy setup allowing to introduce and study non-local effects in conventional models. However, superluminal models of the type (2.6) may appear to be disconnected from theories exhibiting (approximately) conventional behavior. Namely, it seems impossible to couple them to a dynamical sector with conventional properties, such that superluminal effects in this sector would be just a small correction. Indeed, let us consider a conventional free massive scalar field,

$$S = \frac{1}{2} \int d^2x [(\partial\phi)^2 - m^2\phi^2] .$$

In the light cone coordinates the corresponding field equation is first order in “time”  $x^+$ ,

$$2\partial_+ \partial_- \phi + m^2 \phi = 0 . \quad (2.8)$$

Consequently, the initial value problem for a massive scalar field formulated on the set of slices  $x^+ = \text{const}$  contains only one free function instead of two in the conventional Cauchy problem formulated in the coordinates  $(t, x)$ . This, in turn, leads to inequivalent canonical quantization of the theory for the two choices of coordinates<sup>3</sup>. On the other hand, as discussed above, constant  $x^+$  surfaces are the only possible choice of the Cauchy surfaces for the instantaneous systems such as (2.6). So it appears impossible to couple (2.6) to a conventional massive field while keeping instantaneous effects small.

To get around this problem we suggest the following. Let's assume that there is a dynamical vector  $V^\alpha$ , such that its (+) component is non-vanishing. Then one can modify the scalar field action in the following way

$$S = \frac{1}{2} \int d^2x [(\partial\phi)^2 - m^2\phi^2 + \alpha(V^+\partial_+\phi)^2] , \tag{2.9}$$

where  $\alpha > 0$  is a coupling constant. This changes the effective metric for the propagation of the field  $\phi$ , so that the surfaces  $x^+ = \text{const}$  are space-like with respect to the new metric. As a result, the scalar field equation becomes second order in the light cone coordinates, so that the number of degrees of freedom in the initial value problem matches that in the conventional case. Clearly, for this trick to work we need  $V^+ \neq 0$  everywhere for all possible backgrounds. On the other hand, in the limit when  $\alpha$  is small the system (2.9) smoothly approaches the conventional scalar field (2.8). In other words, if both  $\alpha$  and a coupling between the field  $\phi$  and the instantaneous sector are small, an observer coupled just to  $\phi$  may pretend for a while that she lives in a space-time with the conventional causal structure, until effects mediated by the instantaneous sector force her to change her mind.

These considerations lead us again to the question of finding a theory with spontaneous Lorentz violation, such that one always has  $V^+ \neq 0$ . Clearly, as formulated, this proposal may appear superficial as it apparently contradicts to the CMW result on the absence of spontaneous symmetry breaking in two dimensions. We will elaborate on this point later. Remarkably, as we will see now, the theories with  $V^+ \neq 0$  at different choices of parameters by themselves provide quite non-trivial examples of systems with all possible causal ordering in two dimensions including the  $x^+$ -ordered causal structure.

### 3. Description of the model and its causal structure

At the effective field theory level models developing Lorentz breaking condensates of a vector field were studied rather extensively (see e.g. [13, 29–32, 26, 33]). To study a possibility of UV completion of such a model in two space-time dimensions the most appropriate setup to start with is the “Einstein-aether” model of [13, 26]. This is a model describing dynamics of a space-time vector field  $V_\mu$ , subject to a constraint

$$V_\mu V^\mu = 1 . \tag{3.1}$$

---

<sup>3</sup>The situation is different in the case of a massless field where the second free function — the left-moving part of the field — appears as a gauge degree of freedom due to the presence of the first class constraints.

In flat space-time the action of this model is of the form

$$S = \int d^2x \left( -\alpha_1 \partial_\mu V^\nu \partial^\mu V_\nu - \alpha_2 \partial_\mu V^\mu \partial_\nu V^\nu - \alpha_3 \partial_\mu V^\mu \epsilon^{\nu\lambda} \partial_\nu V_\lambda + \lambda (V^\mu V_\mu - 1) \right), \quad (3.2)$$

where  $\lambda$  is a Lagrange multiplier enforcing the constraint (3.1). In space-time dimensions greater than two one may also add the term  $(V^\mu \partial_\mu V^\nu \cdot V^\lambda \partial_\lambda V^\nu)$ ; however, in the two-dimensional case the latter term can be expressed via the terms already present in the action (3.2) [26]. The parity breaking term proportional to  $\alpha_3$  was not included in the original Einstein-aether action [26].

The system (3.2) is a non-linear sigma model with  $SO(1,1)$  symmetry group. This group is one dimensional, so normally the corresponding sigma-model would just describe a free Goldstone field. What makes the situation different in the present case is that the  $SO(1,1)$  group under consideration is a space-time Lorentz group, rather than an internal symmetry group. This leads to non-trivial self-interaction of the Goldstone field. To see this explicitly one solves the constraint (3.1) by introducing the ‘‘rapidity’’ field  $\psi$  directly parametrizing the group manifold,

$$V^\pm = \frac{1}{\sqrt{2}} e^{\pm\psi}. \quad (3.3)$$

The spontaneously broken Lorentz symmetry is realized non-linearly in terms of the Goldstone field  $\psi$ ,

$$\psi(x^+, x^-) \mapsto \psi(e^\gamma x^+, e^{-\gamma} x^-) + \gamma. \quad (3.4)$$

Now the Einstein-aether action (3.2) takes the following form,

$$S = \int dx^+ dx^- \left\{ \frac{1}{g^2} \partial_+ \psi \partial_- \psi + \frac{\beta_{(+)}}{2g^2} (\partial_+ \psi)^2 e^{2\psi} + \frac{\beta_{(-)}}{2g^2} (\partial_- \psi)^2 e^{-2\psi} \right\}, \quad (3.5)$$

where the couplings  $g^2, \beta_{(\pm)}$  are related to  $\alpha_{1,2,3}$  in the following way,

$$g^2 = \frac{1}{2\alpha_1 + \alpha_2}, \quad (3.6)$$

$$\beta_{(\pm)} = \frac{-\alpha_2 \pm \alpha_3}{2\alpha_1 + \alpha_2}.$$

We assume  $g^2 > 0$ . When  $\beta_{(+)} = \beta_{(-)} = 0$  the action (3.5) becomes free and acquires an enhanced symmetry: the space-time Lorentz transformations get disentangled from the shift of the  $\psi$ -field. In the vector field language this case corresponds to  $\alpha_2 = \alpha_3 = 0$ , so that the second and the third terms in (3.2) vanish. This enhances the symmetry of the action to  $SO(1,1) \times SO(1,1)$  with the two factors acting independently on the space-time coordinates and the vector field  $V^\mu$ . Only the second  $SO(1,1)$  factor, which plays the role of an internal symmetry, is spontaneously broken and the sigma-model gives rise to a free theory, as it should be.

Our aim is to use the model (3.5) for exploring the  $x^+$ -ordered causal structure. The Hamiltonian of the system, conjugate to the light-cone variable  $x^+$ , has the form

$$H_{lc} = \int dx^- \left\{ \frac{\beta_{(+)}}{2g^2} (\partial_+ \psi)^2 e^{2\psi} - \frac{\beta_{(-)}}{2g^2} (\partial_- \psi)^2 e^{-2\psi} \right\}. \quad (3.7)$$

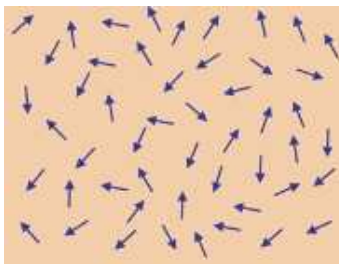
A necessary (but not sufficient) condition for the energy to be positive is  $\beta_{(+)} > 0$ . We adopt this choice in what follows unless stated otherwise.

By power-counting the sigma-model action (3.5) is renormalizable and, as we discuss below, at least for some values of the parameters there are good reasons to believe that the theory can be defined non-perturbatively. Thus, it can be used as a sector giving rise to non-vanishing  $V^+$  in the action (2.9).

As already mentioned above, existence of non-vanishing  $V^+$  seems to be at odds with the CMW theorem. To clarify this issue let us recall that the precise statement of the CMW theorem is that there cannot exist a *symmetry breaking vev*  $\langle V^+ \rangle$ . This does not imply that the notion of a background  $V^+(x^+, x^-)$  completely loses its meaning. In practice, when calculating Green's functions of the system it is often convenient to separate variables in the path integral into a slowly varying background part and fast fluctuations. Then, contributions of the latter into the path integral can be evaluated using the perturbation theory in the background of the smooth part. The subsequent integration over the slow modes can be considered as a sort of “averaging” of the perturbative Green's function over all possible backgrounds. This “averaging” leads to restoration of the symmetry at large distances. Nevertheless, the properties of the short distance physics are captured to large extent by perturbation theory around the symmetry breaking background, cf. [34]. Thus one concludes that when describing the system (2.9) at short time and distance scales one can use perturbation theory around some particular value of  $V^+$  and, as a consequence, the constant  $x^+$  surfaces provide a well-defined set of Cauchy slices. Roughly, at long distances one should “average” over all possible configurations of  $V^+$ , but as  $x^+$  is a good time variable for every individual background  $V^+$ , this should remain the case after making the “average”.

It is worth discussing the physical meaning of the CMW result also from the point of view of the canonical quantization. In the context of condensed matter systems one may think, for instance, about ferromagnetic material. Then, performing a detailed measurement, one finds locally at each point of space a spin pointing in some direction (see figure 5). The CMW statement is that in two dimensions, when one describes the physics of such a system at long distance scales, fluctuations of spins lead to “averaging” of the local spins over all possible directions and result in restoration of the symmetry. Similarly, in the context of the Minkowski space field theory, long range quantum fluctuations result in the symmetry restoration when one considers asymptotic quantities such as the  $S$ -matrix elements. On the other hand, the quantum theory retains many properties of the classical description which operates with the notion of a given field configuration. In particular, in what follows it will be important for us that the quantum theory inherits the causal structure of the classical one (see [28] for a recent discussion).

With these comments in mind let us proceed to the inspection of the model (3.5). One notices that the constant shift of the Goldstone field  $\psi \rightarrow \psi + \gamma$  changes the ratio  $\beta_{(-)}/\beta_{(+)}$  by an arbitrary positive multiplicative factor, so it is only the product  $\beta_{(+)}\beta_{(-)}$  which has a physical meaning. There are four different cases to consider:



**Figure 5:** The net magnetization is averaged out to zero at long distances in a two-dimensional ferromagnetic. Nevertheless, a perturbative description in terms of small excitations around symmetry breaking configurations is appropriate for many local questions.

**3.1**  $\beta_{(+)} = -\beta_{(-)} \equiv \beta$

This choice of parameters explicitly breaks the spatial parity  $x \mapsto -x$ . The light-cone Hamiltonian (3.7) is positive-definite in this case and is minimized at the classical level by configurations  $\psi = const$ . We will see shortly that this theory naturally obeys the  $x^+$ -ordered causal structure. However, to make a comparison with the conventional approach more transparent let us see what happens if one tries to use  $t$  and  $x$  as the time and space variables. Let us fix the perturbative vacuum  $\psi = 0$ . Around this vacuum one finds two modes: left and right movers, with dispersion relations,

$$\omega_{L,R} = v_{L,R} k_{L,R} . \tag{3.8}$$

As a result of parity breaking the two propagation velocities are not equal,

$$v_R = \beta + \sqrt{1 + \beta^2} , \quad v_L = -\beta + \sqrt{1 + \beta^2} . \tag{3.9}$$

Importantly, the right-moving particle propagates superluminally, while the left-moving one subluminally. Taking a different value of  $\psi$  is equivalent to looking at this situation from the point of view of a boosted observer. By choosing an appropriate value of  $\psi$  one may obtain a left-mover propagating in an arbitrary direction in the future light cone. Similarly, with an appropriate value of  $\psi$  one can obtain a right-mover propagating in an arbitrary direction in the right space-like region (see figure 6). Of course, for large enough value of  $\psi$  the right-mover propagates back in the Minkowski time  $t$ , indicating that the well-defined time variable is actually  $x^+$  rather than  $t$ . Using again the intuition that the true long distance correlators can be thought of as a result of “averaging” over different perturbative vacua, we see that the retarded Green’s function of the theory is non-zero in the whole region<sup>4</sup>  $x^+ > 0$ . Thus, this theory enjoys the  $x^+$ -ordered causal structure.

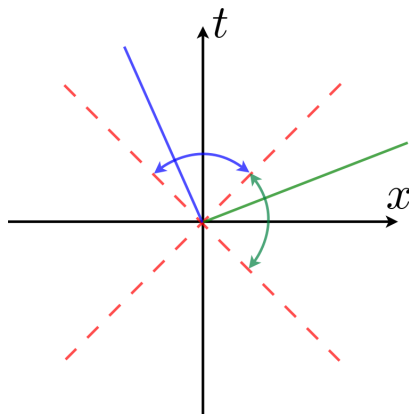
**3.2**  $\beta_{(-)} = 0$

This is also a parity breaking case. The light-cone Hamiltonian (3.7) is again positive, though now the ground state is highly degenerate at the classical level: any configura-

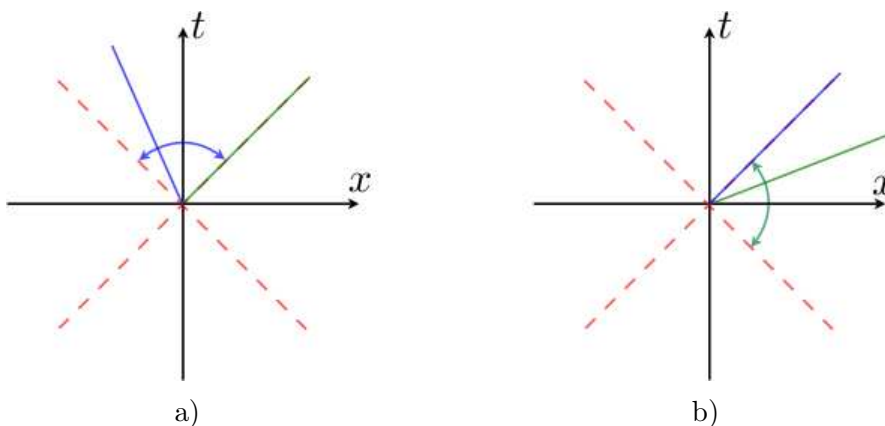
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<sup>4</sup>Note that as a result of the “averaging” one expects the complete Green’s functions of the system to be Lorentz-invariant, in spite of the fact that the perturbative modes have Lorentz-violating dispersion relations (3.8). This is somewhat analogous to the situation in QCD where the asymptotic states are colorless in spite of the fact that the short distance dynamics is described by colored quarks.





**Figure 6:** For  $\beta_{(+)} = -\beta_{(-)}$ , depending on the value of  $\psi$ , the left-mover can propagate in any direction inside the future light cone and the right-mover can propagate in an arbitrary direction in the right space-like region.



**Figure 7:** For  $\beta_{(-)} = 0$ , one of the modes propagates along the light cone. Depending on the sign of the  $\partial_+\psi\partial_-\psi$  term the other mode propagates either inside the future light cone a) or in right space-like region b).

tion  $\psi = \psi(x^-)$  has zero energy. This degeneracy is related to a large symmetry of the action (3.5) with  $\beta_{(-)} = 0$ , discussed in section 5.2.

In spite of the parity breaking, the theory (3.5) with  $\beta_{(-)} = 0$  by itself possesses the standard causal structure. To see this let us again work in terms of the coordinates  $t, x$ . Around the vacuum  $\psi = 0$  one finds two modes: a right-mover propagating with the speed of light, and a subluminal mode propagating with the velocity  $(\beta - 2)/(\beta + 2)$ , where  $\beta \equiv \beta_{(+)}$ . This mode is left-moving for  $\beta < 2$  and right-moving for  $\beta > 2$ . A variation of  $\psi$  is equivalent to a variation of  $\beta$  and results in changing the velocity of the subluminal mode that can propagate in any direction inside of the light cone, depending on the value of  $\psi$  (see figure 7a). Consequently, both at the classical and quantum levels this theory possesses the standard causal structure.

Still, by a slight modification of the setup it is easy to construct theories obeying the

$x^+$ -ordering. One introduces a scalar field  $\chi$  and couples it to  $\psi$  as follows,

$$S = \int dx^+ dx^- \left\{ \frac{1}{g^2} (\partial_+ \psi \partial_- \psi - \partial_+ \chi \partial_- \chi) + \frac{\beta}{2g^2} (\partial_+ \psi)^2 e^{2\psi} + \frac{\beta_\chi}{2g^2} (\partial_+ \chi)^2 e^{2\psi} \right\}, \quad (3.10)$$

where  $\beta_\chi > 0$ . The resulting theory is well-defined, in particular, it has positive light-cone Hamiltonian. In the background  $\psi = 0$  the  $\chi$ -excitations contain a right-moving mode propagating with the speed of light and another propagating superluminally (see figure 7b). The interaction between the fields  $\chi$  and  $\psi$  couples the subluminal ( $\psi$ ) and superluminal ( $\chi$ ) modes leading to a theory with the  $x^+$ -ordered causal structure.

The theory (3.10) has very interesting properties that suggest that this model (possibly with some modifications) may be exactly solvable. This issue is discussed in more detail in section 5.2.

### 3.3 $\beta_{(+)} = \beta_{(-)} \equiv \beta > 0$

This choice preserves the spatial parity  $x \mapsto -x$ . The dispersion relation for linear fluctuations around  $\psi = 0$  background has the form

$$\omega^2 = v^2 k^2 \quad (3.11)$$

with the effective propagation velocity  $v$  given by

$$v^2 = \frac{1 - \beta}{1 + \beta}. \quad (3.12)$$

For  $\beta > 0$  the perturbations are subluminal. According to the arguments above this implies that the model has the standard causal structure. The system is perturbatively stable for  $|\beta| < 1$ . However, the light-cone Hamiltonian (3.7) is unbounded from below. The same is true for the Hamiltonian conjugate to the normal time  $t$ , see eq. (5.13) below. As will be discussed in more detail in section 5.3 this may signal a presence of a non-perturbative instability in the model.

### 3.4 $\beta_{(+)} = \beta_{(-)} \equiv \beta < 0$

Note that eq. (3.12) implies that the system is perturbatively stable also for  $0 > \beta > -1$ . In this case the velocity (3.12) is greater than 1, i.e. the perturbative modes are superluminal. This last case is rather different from all the previous ones, and is somewhat outside the main line of the current paper. Indeed, existence of the superluminal modes propagating in both directions cannot be reconciled with the restoration of Lorentz symmetry required by the CMW theorem if the theory has any fixed causal structure. One possible attitude at this point could be that this model is inconsistent.

However, in principle, quantum field theories without fixed causal structure might exist (gravity being one example). Below, we try to take an optimistic viewpoint that the theory (3.5) makes sense also in this case and see where this assumption may lead us. The bulk of the discussion is postponed until section 5.4. Here we limit ourselves to the observation that at the classical level the theory admits a well-posed initial value

problem provided one uses surfaces that are space-like with respect to the effective metric determining propagation of small perturbations (see eq. (4.2) below) to define the set of Cauchy slices. Note that the argument of section 2.2 about the absence of closed time-like curves applies to the model at hand as well, so that the initial value problem is well-defined not only locally but also globally.

#### 4. One-loop running

To gain more insight into the dynamics of the Einstein-aether theory it is instructive to study the one-loop running of the parameters  $g^2, \beta_{(\pm)}$ . This can be done in a uniform way for all the choices of  $\beta_{(+)}, \beta_{(-)}$ .

To get around the infrared divergencies it is convenient to make use of the background field method (cf., e.g., the classic calculation [36]). In this approach one decomposes the field  $\psi$  into a long-distance part  $\psi_0$  with momenta smaller than some value  $\tilde{\Lambda}$  and the fluctuating part  $\pi$  with momenta greater than  $\tilde{\Lambda}$ . The fluctuations  $\pi$  are then integrated out for the range of momenta between  $\tilde{\Lambda}$  and the UV cutoff  $\Lambda > \tilde{\Lambda}$ . The Lorentz symmetry (3.4) implies that as a result of this procedure one will obtain the action for  $\psi_0$  of the form (3.5) with couplings evaluated at the scale  $\tilde{\Lambda}$  plus higher dimensional irrelevant interactions.

At the one loop level we need only the part of the action quadratic in  $\pi$  (given that we are not interested in keeping track of the irrelevant operators). The latter is conveniently presented in the following form,

$$S^{(2)}[\psi_0, \pi] = \frac{\sqrt{1 - \beta_{(+)}\beta_{(-)}}}{2g^2} \int d^2x \sqrt{|G|} (G^{\mu\nu} \partial_\mu \pi \partial_\nu \pi - \mathcal{M}^2 \pi^2) , \quad (4.1)$$

where the metric  $G^{\mu\nu}$  is given by

$$G^{++} = \beta_{(+)} e^{2\psi_0} , \quad G^{--} = \beta_{(-)} e^{-2\psi_0} , \quad G^{+-} = 1 \quad (4.2)$$

and the mass  $\mathcal{M}^2$  is

$$\mathcal{M}^2 = 2\beta_{(+)} e^{2\psi_0} ((\partial_+ \psi_0)^2 + \partial_+^2 \psi_0) + 2\beta_{(-)} e^{-2\psi_0} ((\partial_- \psi_0)^2 - \partial_-^2 \psi_0) . \quad (4.3)$$

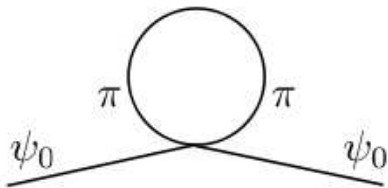
The one loop contribution to the action  $\delta S_{1-loop}$  is given by

$$e^{i\delta S_{1-loop}[\psi_0]} = \int \mathcal{D}\pi e^{iS^{(2)}[\psi_0, \pi]} , \quad (4.4)$$

where the integral is taken over modes with momenta between  $\tilde{\Lambda}$  and  $\Lambda$ . Note that  $\delta S_{1-loop}$  is a diff invariant functional of the metric  $G^{\mu\nu}$  and of the mass  $\mathcal{M}^2$ . Therefore, it has the form

$$\delta S_{1-loop}[\psi_0] = \int d^2x \sqrt{|G|} (c_0 - c_1 \mathcal{M}^2 + \dots) , \quad (4.5)$$

where dots stand for higher dimensional irrelevant operators and  $c_0, c_1$  are some coefficients. We do not include the Einstein-Hilbert term  $\sqrt{|G|} R_G$  into (4.5) as this term is a total derivative in two dimensions. Next,  $\sqrt{|G|}$  is just a constant that doesn't depend on  $\psi_0$ .



**Figure 8:** The diagram contributing to one-loop running.

Consequently, the only non-trivial running is related to the second term in (4.5). By comparing the expression (4.5) to the original action (3.5) we find that  $g^2$  remains constant at the one-loop level. To reconstruct the one-loop RG equations for  $\beta_{(\pm)}$  we need to calculate the coefficient  $c_1$ . It is given by the diagram of figure 8,

$$c_1 = \frac{\sqrt{1 - \beta_{(+)}\beta_{(-)}}}{2g^2} \int \frac{d^2p}{(2\pi)^2} \langle \pi(p)\pi(-p) \rangle, \quad (4.6)$$

where the  $\pi\pi$  propagator is determined by the local value of the  $G^{\mu\nu}$  metric,

$$\langle \pi(p)\pi(-p) \rangle = \frac{ig^2}{2p_+p_- + \beta_{(+)}e^{2\psi_0}p_+^2 + \beta_{(-)}e^{-2\psi_0}p_-^2 + i\epsilon}. \quad (4.7)$$

Note that the  $i\epsilon$ -prescription in (4.7) is chosen to be compatible with the  $x^+$ -ordered causal structure. It agrees with the conventional time-ordering in the cases considered in sections 3.2, 3.3.

To evaluate the integral (4.6) it is convenient to make an analytic continuation

$$p_{\pm} \mapsto \pm \frac{p}{\sqrt{2}} e^{\pm i\phi}.$$

This continuation is nothing else as the standard Wick rotation  $p_0 \mapsto -ip_0$ . Note that due to non-standard structure of the propagator (4.7) the integration contour generically hits poles in the complex  $(p_+, p_-)$  space during the Wick rotation, and one should be careful to properly include contributions of these poles. This complication does not arise for sufficiently small values of the Lorentz-violating parameters  $\beta_{(\pm)}$ . Thus, the simplest way to perform the integral is to evaluate it for small  $\beta_{(\pm)}$  and then analytically continue to larger values of  $\beta_{(\pm)}$ . One obtains

$$c_1 = \frac{\sqrt{1 - \beta_{(+)}\beta_{(-)}}}{8\pi^2} \int_{\tilde{\Lambda}}^{\Lambda} \frac{dp}{p} \int_0^{2\pi} d\phi \left[ 1 - \frac{\beta_{(+)}e^{2\psi_0}}{2} e^{2i\phi} - \frac{\beta_{(-)}e^{-2\psi_0}}{2} e^{-2i\phi} \right]^{-1} = \frac{1}{4\pi} \log \frac{\Lambda}{\tilde{\Lambda}}. \quad (4.8)$$

Comparing this result to the original action for  $\psi$  (3.5) we find

$$\beta_{(\pm)}(\tilde{\Lambda}) = \beta_{(\pm)}(\Lambda) + \frac{g^2\beta_{(\pm)}(\Lambda)}{\pi\sqrt{1 - \beta_{(+)}\beta_{(-)}(\Lambda)}} \log \frac{\Lambda}{\tilde{\Lambda}}.$$

Equivalently, couplings  $\beta_{(\pm)}$  satisfy the following RG equations

$$\frac{d\beta_{(\pm)}}{d\log\Lambda} = -\frac{g^2\beta_{(\pm)}}{\pi\sqrt{1-\beta_{(+)}\beta_{(-)}}}, \quad (4.9)$$

These equations (4.9) imply that the ratio  $\beta_{(-)}/\beta_{(+)}$  remains constant while the product  $\beta_{(+)}\beta_{(-)}$  flows to zero in the UV. Consequently, at high energies both Lorentz violating parameters  $\beta_{(\pm)}$  flow to zero and the theory is asymptotically free.<sup>5</sup>

At this point it is worth mentioning that the Euclidean SO(2) version of the model (3.5) with  $\beta_{(+)} = \beta_{(-)}$ , whose action is obtained by substituting  $\psi \rightarrow i\psi$  and  $g^2 \rightarrow -g^2$ , describes nematic liquid crystals [35]. The corresponding RG equation agrees with (4.9) after reversing the sign of  $g^2$ . However, this changes the sign of the beta-function, so that in the liquid crystal case  $\beta_{(\pm)}$  are irrelevant parameters. Thus the physics of two models is very different.

Note that the RG flow (4.9) is derived in the leading order in  $g^2$  but to all orders in  $\beta_{(\pm)}$ . There is no reason to expect that the coupling  $g^2$  remains constant at the higher loop level.<sup>6</sup> Still, the fact that the model with  $\beta_{(\pm)}$  is free at arbitrary values of  $g^2$  implies that the  $g^2$  running is proportional to  $\beta$ . This means that the theory will remain asymptotically free once the RG flow enters into the region of small  $g^2$  and  $\beta$ . A straightforward analysis of possible two-loop corrections shows that this always happens if the RG flow starts at a given scale from any value of  $\beta$  and sufficiently small  $g^2$ . Nevertheless, it is possible that the structure of the RG flow changes qualitatively at larger (but, perhaps, still perturbative) values of  $g^2$ . It would be of interest to perform the two-loop calculation in this model to see if this is the case.

## 5. Nonperturbative considerations

Let us now turn to the non-perturbative aspects of the Einstein-aether model. These are quite different in the four cases listed in section 3.

### 5.1 Case $\beta_{(+)} = -\beta_{(-)} \equiv \beta$

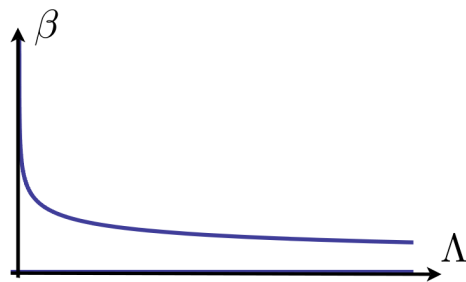
As already noticed above, this case obeys the  $x^+$ -ordered causal structure. Thus, it is natural to interpret  $x^+$  and  $x^-$  as time and space variables. The Wick rotation  $x^+ \rightarrow -ix^+$  results in the Euclidean action with a positive real part, strongly suggesting that the theory is consistent at the non-perturbative level. Also, the light-cone Hamiltonian (3.7) describing evolution in  $x^+$  is positive definite in this case, so the system is stable, at least at the semiclassical level.

The RG equation (4.9) implies that  $\beta$  evolves from zero in the UV to infinity in the IR (see figure 9). This does not pose any problem as the perturbation theory works at all

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<sup>5</sup>Strictly speaking, this statement is correct only for the cases when both  $\beta_{(\pm)}$  are non-zero. In the  $\beta_{(-)} = 0$  case  $\beta_{(+)}$  also flows to zero in the UV, however running of  $\beta_{(+)}$  does not have physical meaning in this case, as the value of  $\beta_{(+)}$  can be changed by a shift of  $\psi$ .

<sup>6</sup>Except the case  $\beta_{(-)} = 0$  where  $g^2$  is not renormalized in all orders of the perturbation theory, see below.



**Figure 9:** One-loop running for  $\beta_{(+)} = -\beta_{(-)}$ .

values of  $\beta$ . In principle, the IR behavior could change at the higher loop level, where we expect  $g^2$  coupling to run as well. Then  $g^2$  could blow up at some IR scale as a result of this running. However, at least in some range of parameters, this doesn't happen. Indeed, in the limit when  $\beta$  is large one can neglect the first term in the action and arrive at the action of the form,

$$S = \frac{1}{2\kappa^2} \int dx_+ dx_- \left[ (\partial_+ \psi)^2 e^{2\psi} - (\partial_- \psi)^2 e^{-2\psi} \right], \quad (5.1)$$

where  $\kappa^2 = g^2/\beta$ . This theory is weakly coupled at small  $\kappa^2$  and the one-loop RG equation for  $\kappa^2$  can be obtained from (4.9) by taking the limit  $\beta \rightarrow \infty$ ,

$$\frac{d\kappa}{d \log \Lambda} = \frac{\kappa^3}{2\pi}. \quad (5.2)$$

This equation shows that the theory is free in the IR, so that no dynamical IR scale is generated.

In the UV the theory flows to the asymptotically free Lorentz symmetric point  $\beta = 0$ . Consequently, this model by itself (even without introducing couplings to additional fields as in (2.9)) provides an example of a theory possessing the  $x^+$ -ordered causal structure in such a way, that in a certain regime an observer may be tricked into thinking that the causal structure is the standard one. Indeed, the smallness of  $\beta$  in the UV implies that results of short distance experiments can be with a good accuracy described in terms of the conventional causal structure. To figure out that the actual causal structure is different, an observer should either wait for a long enough time (and/or perform an experiment in a large enough volume), or make precision measurements of the  $\beta$  suppressed effects at short distances. Note, that these effects, when interpreted within the standard causal structure, appear completely non-local (and even acausal). Also the true causal structure features instantaneous interactions, that definitely qualify as a non-local effect. So this theory provides an example of a field theoretical model giving rise to the genuinely non-local physics, in spite of a benign looking and local action. A somewhat similar behavior is exhibited by four dimensional effective field theories discussed in [18]. Interestingly, here one observes this in, as far as we can tell, a fully consistent microscopic quantum theory.

## 5.2 Case $\beta_{(-)} = 0$

We are going to see that this case has a number of interesting properties that potentially allow to move on rather far in the non-perturbative analysis of the model and even give rise to a hope to obtain a solvable model with  $x^+$ -ordered causal structure. The action has the following form

$$S = \int dx^+ dx^- \left\{ \frac{1}{g^2} \partial_+ \psi \partial_- \psi + \frac{\beta}{2g^2} (\partial_+ \psi)^2 e^{2\psi} \right\}. \quad (5.3)$$

It has obvious similarities with that of the Liouville theory. Like in the Liouville case,  $\beta$  is not a physically meaningful parameter any longer: by making the field shift

$$\psi \rightarrow \psi + \text{const}$$

one can change the value of  $\beta$  by an arbitrary positive factor.

An important property of the Liouville theory is its conformal invariance. The action (5.3) is not invariant under the full conformal group due to the presence of  $(\partial_+ \psi)^2$  in the second term. Still, at the classical level, (5.3) is invariant under right mover's subgroup of the conformal group and under the global part of the left mover's subgroup,

$$\begin{aligned} x^+ &\mapsto f(x^+), & x^- &\mapsto g(x^-), \\ \psi(x^+, x^-) &\mapsto \psi(f(x^+), g(x^-)) - \frac{1}{2} \log f'(x^+) + \frac{1}{2} \log g'(x^-), \end{aligned} \quad (5.4)$$

where

$$f = \frac{ax^+ + b}{cx^+ + d}, \quad ad - bc = 1,$$

and  $g$  is a generic function. As in the Liouville case, the field  $\psi$  transforms inhomogeneously under the conformal transformations (5.4). We are not able to say at the moment whether the whole symmetry (5.4) can be preserved at the quantum level.

There is yet another property that the model (5.3) shares with the Liouville theory. Namely, a simple power counting shows that in the Liouville theory (and, more generally, in any two dimensional field theory with non-derivative interactions) the only UV divergences in Feynmann diagrams are due to loops where an internal line has both ends on the same vertex (figure 10). As a result, all UV divergencies in these theories can be removed by the free field normal ordering. In the Liouville case this allows to perform the canonical quantization of the theory and to establish a large number of properties (such as the conformal invariance) at the non-perturbative level [38].

The action (5.3) involves derivative interactions, so the naive power counting would suggest that there should be divergences that cannot be removed by the free field normal ordering; for instance, one would expect the right diagram in figure 10 to diverge. As we are going to show this conclusion is wrong, and there is a simple argument involving a modified power counting that shows that the action (5.3) can also be renormalized just by the free field normal ordering. The actual statement is even stronger. Namely, we will show that any two dimensional action of the form

$$S = \int dx^+ dx^- [\partial_+ \psi \partial_- \psi + U(\partial_+, \psi)] \quad (5.5)$$



**Figure 10:** UV divergencies originated from the diagram on the left are removed by the normal ordering. The right diagram is finite in two-dimensional theories with non-derivative interactions, as well as in a more general class of theories (5.5).

is renormalizable by the free field normal ordering. Here  $U$  is an arbitrary function of the field  $\psi$  and its derivatives in the  $x^+$  direction. This result is somewhat unexpected, given that the naive power counting suggests that, generically, theories (5.5) are non-renormalizable. To prove that they are actually renormalizable, and that all divergences can be removed by normal ordering, it is convenient to use the language of the Wilsonian RG. Let us assume that with a certain cutoff scale  $\Lambda$  the action takes form (5.5) and let us write the (generalized) potential function  $U$  in the following form

$$U = \sum \frac{\alpha_j}{M^{j-2}} U_j(\partial_+, \psi), \quad (5.6)$$

where  $U_j$  stand for all terms in  $U$  that have exactly  $j$  derivatives,  $M$  is the parameter of dimension mass that gives the correct dimensionality to all the terms,  $\alpha_j$  are dimensionless parameters, and the functions  $U_j$  don't contain any dimensionful parameters in them. Note that the action (5.5) is invariant under Lorentz transformations

$$\psi(x^+, x^-) \mapsto \psi(e^\gamma x^+, e^{-\gamma} x^-) \quad (5.7)$$

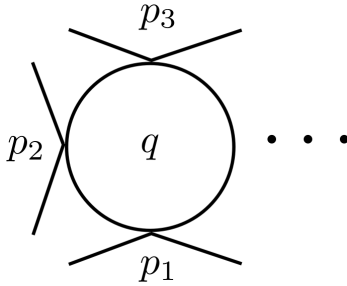
if  $\alpha_j$  are considered as spurion fields transforming as

$$\alpha_j \mapsto e^{-j\gamma} \alpha_j. \quad (5.8)$$

Let us see what restrictions this symmetry implies on the form of the action obtained by integrating out all modes between  $\Lambda$  and some lower scale  $\tilde{\Lambda}$ . Note, that the special form of the action (5.5) is not protected by any symmetry, so that in general the effective action at the scale  $\tilde{\Lambda}$  will contain terms with  $\partial_-$  derivatives as well. However, the symmetry (5.7), (5.8) and dimensional considerations impose strong restrictions on the dependence of various terms on the UV cutoff  $\Lambda$ . Indeed, let us consider a term proportional to a product of  $k \geq 1$  coupling constants  $\alpha_{j_1} \cdot \dots \cdot \alpha_{j_k}$  in the effective Lagrangian. The symmetry (5.7), (5.8) fixes the difference between the number of  $\partial_+$  and  $\partial_-$  derivatives in this term. Moreover, the dependence of the action (5.5) on  $M$  implies that this term contains  $M^{J-2k}$ , where  $J = j_1 + \dots + j_k$ , in the denominator. Requiring that this term has mass dimension 2 one obtains its possible form,

$$\frac{\alpha_{j_1} \cdot \dots \cdot \alpha_{j_k}}{M^{J-2k}} \frac{\partial_+^J}{\Lambda^{2k-2}} \left( \frac{\partial_+ \partial_-}{\Lambda^2} \right)^n \left( \log \frac{\Lambda}{\tilde{\Lambda}} \right)^m. \quad (5.9)$$





**Figure 11:** A general one-loop graph.

We see that the only possible UV divergences are associated with the terms containing no  $\partial_-$  ( $n = 0$ ), and proportional to a single coupling constant  $\alpha$  ( $k = 1$ ). These are the logarithmic divergencies coming from the loops where an internal line has both ends on the same vertex as in the left diagram in figure 10, i.e. exactly those removed by normal ordering.

Given a somewhat formal character of the above argument it is instructive to see how it works at the diagrammatic level. Let us consider a general one loop diagram as shown in figure 11. After the standard Wick rotation this diagram gives rise to the Feynmann integral of the form

$$I = \int \frac{d^2 q q_+^n}{\prod_i ((q + p_i)^2 + \mu^2)}, \quad (5.10)$$

where  $q$  is the internal loop momentum,  $p_i$  is the external momentum coming from the  $i$ -th vertex, and we introduced a mass  $\mu^2$  as an IR regulator. The only difference between this integral and the usual one obtained in the absence of derivative interactions, is the  $q_+^n$  factor in the numerator coming from  $\partial_+$ 's in the interaction vertices. Naively, this factor changes the converging properties of the integral by introducing an additional  $n$ -th power of the momentum in the numerator. To see why this naive counting is wrong, it is convenient to switch to the polar coordinates. Then the Feynmann integral (5.10) takes the following form,

$$I = \int_0^\Lambda dq q^n \int_0^{2\pi} \frac{d\phi e^{in\phi}}{\prod_i (q^2 + p_i^2 + \mu^2 + 2qp \cos(\phi - \phi_i))}, \quad (5.11)$$

where

$$q_\pm = q e^{\pm i\phi}, \quad p_{i\pm} = p_i e^{\pm i\phi_i}.$$

The point is that  $\partial_+^n$  brings in not only a factor  $q^n$  in the numerator, but also a phase factor  $e^{in\phi}$ . To see the impact of this factor one Taylor expands the denominator in (5.11) in powers of  $\cos(\phi - \phi_i)$ . At large  $q$  each power of  $\cos(\phi - \phi_i)$  brings in one factor of  $q^{-1}$ . To compensate the phase  $e^{in\phi}$  and make the angular integral non-zero one needs at least  $n$  of such factors. As a result, after  $\phi$  integration one obtains the integral over  $q$  with the same UV behavior as in a theory without derivative interactions. Consequently, the divergences can be removed by the free field normal ordering.

Clearly, these arguments are perturbative in nature and they do not change the fact that if some of the coefficients  $\alpha_n$  with  $n > 2$  is non-zero (i.e., if the theory is naively non-renormalizable) the derivative expansion breaks down in the UV at the scale  $M$  and the theory becomes strongly coupled at that scale. Nevertheless, given that it is possible to renormalize the theory by normal ordering, one may hope to get around this strong coupling and extract some information non-perturbatively. It would be especially interesting to consider the following family of Lorentz invariant Lagrangians,

$$S = \int dx^+ dx^- \left[ \partial_+ \psi \partial_- \psi + M^2 U(M^{-1} \partial_+ e^\psi) \right] \quad (5.12)$$

generalizing the Einstein-aether action (5.3).

An immediate consequence of the fact that (5.3) is renormalized by normal ordering is that the coupling  $g^2$  remains constant to all orders in the perturbation theory and the parameter  $\beta$  is renormalized multiplicatively. Given that any particular value of  $\beta$  has no physical meaning and can be changed by the shift of  $\psi$  we conclude that the classical scale symmetry (which corresponds to the choice  $f = \alpha x^+$ ,  $g = \alpha x^-$  in (5.4)) is preserved also at the quantum level.

The arguments about renormalizability by normal ordering presented in this section can be easily generalized to the case of Lagrangians containing several fields. In particular, they are applicable to the model (3.10). Besides, it is straightforward to check that the model (3.10) at the classical level enjoys the “half-conformal” symmetry (5.4) with the field  $\chi$  transforming as a usual scalar boson. Therefore, the model (3.10) provides an example of a theory that obeys the  $x^+$ -ordered causal structure, is classically invariant under the symmetry (5.4) and can be renormalized by normal ordering. These properties suggest that the model (possibly, with some modifications) may be exactly solvable.

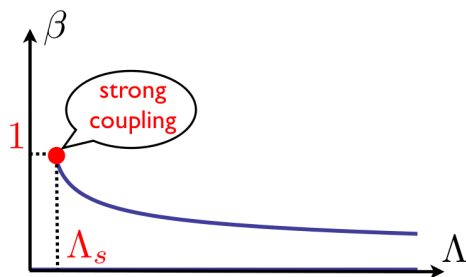
### 5.3 Case $\beta_{(+)} = \beta_{(-)} = \beta > 0$

This case is parity symmetric and consequently it is not directly related to the main subject of the present paper, i.e. the investigation of the  $x^+$ -ordered causal structure that is incompatible with parity. Still, for completeness, we discuss this case as well.

Let us start with the analysis of the RG equation (4.9) in this case. As already mentioned,  $\beta$  runs to zero in the UV. In the IR it approaches unity at some scale  $\Lambda_c$  (see figure 12). Naively, eq. (3.12) implies that the theory becomes unstable when one goes to even longer distance scale, where  $\beta > 1$ . What actually happens is that the model becomes strongly coupled at the scale  $\Lambda_c$ . To see the origin of the strong coupling, note that at the scale  $\Lambda_c$  where  $\beta$  approaches unity, the  $\pi\pi$  propagator in the background  $\psi_0$  degenerates

$$\langle \pi\pi \rangle = \frac{ig^2}{(p_+ + p_-)^2} = \frac{ig^2}{2p_0^2}.$$

This implies that the propagation velocity of the  $\pi$  modes turns zero at the scale  $\Lambda_c$ . Equivalently, the gradient term  $(\partial_x \pi)^2$  drops out from the quadratic action for  $\pi$ . Consequently, one is forced to take into account the higher dimensional operators in the effective action, such as  $(\partial_x^2 \pi)^2$ , and the perturbation theory breaks down.



**Figure 12:** One-loop running for  $\beta_{(+)} = \beta_{(-)}$ .

Interestingly, this behavior is typical also for Lorentz breaking effective theories in four dimensions [10, 11]. There is an important difference, however. In four dimensions, inclusion of the leading higher derivative contributions to the quadratic action is sufficient to get around the breakdown of the perturbation theory. As a result one obtains a low energy effective theory of weakly interacting Goldstone bosons, albeit with an exotic dispersion relation  $\omega^2 \propto k^4$ . This theory is healthy, provided the sign in front of the  $(\partial_x^2 \pi)^2$  operator in the action is negative.

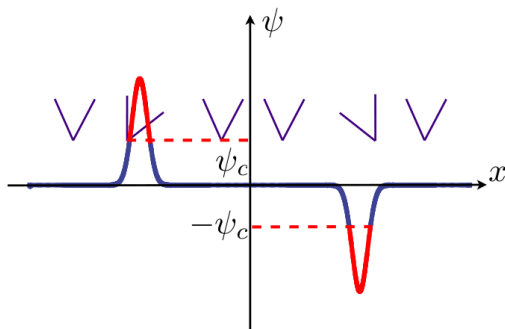
To see what happens instead in two dimensions, let us recall that the modified dispersion relation changes the power counting for the scaling dimensions of the operators in the effective theory [37, 10]. In the case  $\omega^2 \propto k^4$ , if one takes the scaling dimension of energy to be equal to 1, the scaling dimension of the spatial momentum is equal to 1/2. It is straightforward to check that the scaling dimension of the field  $\psi$  becomes then  $(D - 3)/4$ , where  $D$  is the space-time dimensionality. It is negative at  $D = 2$  so that the effective field theory acquires an infinite number of relevant operators (a simple example being  $(\partial_x \psi)^4$  coming from  $(\partial_+ \psi \partial_- \psi)^2$ ). Consequently, the derivative expansion is totally doomed and the theory is strongly coupled at  $\Lambda_c$ .

This behavior of the model — the RG running from asymptotic freedom in UV to strong coupling at a finite scale in IR — is similar to that of other two-dimensional sigma models, such as, e.g., the  $O(N)$  model [36]. Following this analogy it is natural to expect that the strong coupling leads to generation of a mass gap  $\sim \Lambda_c$  and restoration of the Lorentz symmetry at this scale. On the other hand, the physics at short distances can be accurately described by a perturbation theory around the symmetry breaking semiclassical vacuum  $\psi = 0$  (or  $\psi = const$ —all these vacua are related by boosts).

However, this physical picture should be taken with a grain of salt. The reason is that, as mentioned in section 3, the light-cone Hamiltonian of the system is unbounded from below which may indicate a non-perturbative instability. Let us investigate this problem using the conventional coordinates  $(t, x)$ . The Noether procedure gives rise to the following expressions for the conserved energy  $H$  and momentum  $P$  of the  $\psi$  field,

$$H = \frac{1}{2g^2} \int dx \left\{ (1 + \beta \cosh 2\psi) (\partial_t \psi)^2 + (1 - \beta \cosh 2\psi) (\partial_x \psi)^2 \right\} , \quad (5.13)$$

$$P = \frac{1}{g^2} \int dx \left\{ (1 + \beta \cosh 2\psi) \partial_t \psi \partial_x \psi + \beta \sinh 2\psi (\partial_x \psi)^2 \right\} . \quad (5.14)$$



**Figure 13:** An example of a static field configuration with zero energy and momentum. Spikes with  $|\psi| > \psi_c$  have negative energy density that cancels positive energy outside the spikes. The total momentum is zero because the field configuration is odd under  $x \rightarrow -x$  parity. The causal cones of the effective metric for  $\psi$  perturbations are shown schematically. Each spike acts as an event horizon for  $\psi$  perturbations.

Note that the gradient energy becomes negative for large field values suggesting that the semiclassical vacuum is indeed unstable at the non-perturbative level. Let us see what are the field configurations the vacuum can decay to, namely let us construct sample field configurations that have a form of localized excitations around  $\psi = 0$  vacuum at  $t = 0$  and carry zero energy and momentum. One can set the time derivative of the field to zero at  $t = 0$ ,

$$\partial_t \psi(0, x) = 0, \tag{5.15}$$

so that the kinetic energy and the first term in the expression (5.14) for the spatial momentum vanish. Then a simple example of a localized field configuration with zero energy is a high enough spike, such that the negative gradient energy in the region with  $\psi > \psi_c$ , where

$$\psi_c = \frac{1}{2} \cosh^{-1}(1/\beta), \tag{5.16}$$

compensates for the positive gradient energy in the region with  $\psi < \psi_c$  (see figure 13). Note that in principle a spike need not be narrow and may be more like a bump; all what one needs is a region with  $\psi > \psi_c$ , such that the spatial field gradient does not vanish everywhere inside this region and compensates for the positive gradient energy outside. Finally, note that it is impossible for a single spike (or bump) to be spontaneously created from the vacuum as it carries a non-zero spatial momentum, coming from the second term in (5.14). However, it is straightforward to make the total momentum vanish by introducing the second spike, where the field  $\psi$  takes large *negative* values,  $\psi < -\psi_c$ . For instance, any field configuration satisfying (5.15) and odd with respect to spatial reflections,  $\psi(-x) = -\psi(x)$ , has vanishing spatial momentum.

To understand better the physical interpretation of these spikes, let us study the structure of the causal cones of the effective metric (4.2) describing propagation of the scalar fluctuations in the presence of a spike. At  $\psi = 0$  this is just a symmetric cone of a subluminal particle that propagates with velocity (3.12) in both directions. The transformation rule (3.4) implies that taking  $\psi > 0$  is equivalent to looking at the same situation from the

point of view of an observer who moves left with the rapidity equal to  $\psi$ . Consequently, at  $\psi > 0$  the causal cone of the  $\psi$  particles is bended in the right direction. At  $\psi > \psi_c$  this bending becomes so strong that both “left” and “right” moving waves start propagating in the right direction (see figure 13). Consequently, the negative (or zero) energy spike acts as a horizon for the  $\psi$  signals.

It will be interesting to calculate the rate of the instability towards producing field configurations of the type shown in figure 13 out of the vacuum. Presumably, at small couplings this decay is described by a semiclassical bounce solution, however, we have not been able to find this solution so far. Still, assuming that such a solution exists, let us speculate on possible implications of the vacuum decay process. Due to classical scale invariance of the Einstein-aether action (3.5), the bounce solution is not unique, rather there is a whole continuous family of solutions of a different size  $R$ . On dimensional grounds the vacuum decay rate has the form

$$\Gamma \sim \int dR R^{-3} e^{-S_b(R)},$$

where the dependence of the bounce action  $S_b$  on  $R$  arises through the RG running of the couplings. The theory is free in the UV, so that  $S_b \rightarrow \infty$  as  $R \rightarrow 0$ . However, because of the  $R^{-3}$  prefactor, it still may happen that the total decay rate is UV dominated and diverges. This would imply an inconsistency of the model. A more interesting option is that  $\Gamma$  is saturated at large values of  $R$ . Then, creation of the field configurations as in figure 13 may provide a natural mechanism for restoration of the Lorentz symmetry in the IR. Still, it is not clear if the system ends up in a well-defined vacuum state or exhibits a runaway behavior.

To summarize, our analysis of the parity preserving case,  $\beta_{(+)} = \beta_{(-)} > 0$ , is somewhat inconclusive. This model can either be plagued by non-perturbative instabilities or can be a consistent quantum field theory analogous to the conventional two-dimensional sigma-models with strong coupling and gap generation in the IR.

#### 5.4 Case $\beta_{(+)} = \beta_{(-)} \equiv \beta < 0$

Naively one may expect that this model suffers from a non-perturbative instability similar to that present in the  $\beta_{(+)} = \beta_{(-)} > 0$  case due to the fact that the coefficient in front of the  $(\partial_t \psi)^2$  term in the Hamiltonian (5.13) is not positive definite. However, the above arguments cannot be generalized to the present case. Indeed, it is the coefficient in front of the kinetic term in the Hamiltonian (5.13) that becomes negative now, so that it is impossible to find a configuration with negative energy while keeping  $\partial_t \psi = 0$ . More importantly, it is straightforward to check that when the field  $\psi$  exceeds the critical value (5.16), so that there is a chance to have negative energy, the causal cone of the effective metric (4.2) is bended back in time, so that the  $t = \text{const}$  surface is not space-like with respect to this metric any more and cannot be used to define the initial Cauchy data.

One may try to define a field configuration with vacuum quantum numbers on a surface with a more complicated shape, such that it is space-like with respect to the effective metric (4.2) even for  $\psi > \psi_c$ . However, it is straightforward to see that this is impossible.

Indeed, if a slice has a normal vector

$$n_\alpha = (1, \theta)$$

at some point, then in order for this vector to be timelike with respect to the effective metric (4.2) the following inequality should hold

$$G^{\alpha\beta} n_\alpha n_\beta > 0, \tag{5.17}$$

where  $G^{\alpha\beta}$  is the effective metric (4.2). In  $(t, x)$  coordinates it reads

$$G^{\alpha\beta} \partial_\alpha \partial_\beta = (1 + \beta \cosh 2\psi) \partial_t^2 + 2\beta \sinh 2\psi \partial_t \partial_x + (-1 + \beta \cosh 2\psi) \partial_x^2. \tag{5.18}$$

On the other hand, the Noether energy density evaluated at this point can be presented in the following form

$$T_0^\alpha n_\alpha = \frac{1}{2g^2} \left( G^{\alpha\beta} n_\alpha n_\beta (\partial_t \psi)^2 - G^{xx} (\partial_x \psi - \theta \partial_t \psi)^2 \right), \tag{5.19}$$

where

$$T_\beta^\alpha = \frac{1}{g^2} \left( G^{\alpha\nu} \partial_\nu \psi \partial_\beta \psi - \frac{1}{2} \delta_\beta^\alpha G^{\mu\nu} \partial_\mu \psi \partial_\nu \psi \right) \tag{5.20}$$

is the Noether energy-momentum tensor of the Einstein-aether model. Consequently, the energy density (5.19) is non-negatively definite on a spacelike slice as soon as  $G^{xx} < 0$ , which is true for  $\beta < 0$ . This suggests that at least at the semiclassical level the vacuum is stable.

These considerations appear encouraging. Still there remains a question how to make sense of a quantum theory without a fixed causal structure. Note that the same question arises in the case of quantum gravity.<sup>7</sup> In our case the only possibility we see (apart from the option that the theory is pathological), is that the spatial parity in spite of being unbroken at the level of the action is nevertheless broken spontaneously.<sup>8</sup> Indeed, this opens a room for the vacuum retarded Green's function to be Lorentz-invariant but parity violating. The two vacua related by spatial parity would be characterized by the  $x^+$ -ordered and  $x^-$ -ordered retarded Green's functions respectively. Still the absence of the fixed causal structure would manifest itself in the existence of more general quantum states that can be thought of as an admixture of finite size domains with different causal ordering.

It definitely remains unclear whether it may be possible to make sense of such a situation. However, if the answer is yes, this would be a prototype of a higher dimensional theory where superluminal propagation arises due to spontaneous breaking of Lorentz symmetry as discussed in section 2.1. Spontaneous breaking of the continuous Lorentz symmetry is impossible in 2d, however, in this case it is enough just for the spatial parity — a discrete symmetry — to be spontaneously broken.

To conclude this discussion and to avoid a possible confusion let us stress that this controversial case is totally unrelated from our previous analysis of theories with fixed instantaneous causal structure. The consistency of the latter theories will not be affected by the eventual conclusion on whether  $\beta < 0$  theory is pathological or not.

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<sup>7</sup>One can push the analogy between gravity and the  $\beta < 0$  theory even further: in both cases it is problematic to define a consistent path integral due to the absence of a positive definite Euclidean action.

<sup>8</sup>Recall that the spontaneous breaking of a discrete symmetry is not forbidden by the CMW theorem.

## 6. Coupling to gravity

Let us discuss now what happens when the  $SO(1,1)$  sigma-model studied above is coupled to gravity. Such a coupling is straightforward to introduce using the Einstein-aether form of the action (3.2). One obtains

$$S_{gr} = -\frac{1}{2\pi\kappa} \int d^2x \sqrt{-g} (R + \mu^2) + S_{EA}(g_{\mu\nu}, V_\mu), \quad (6.1)$$

where  $S_{EA}(g_{\mu\nu}, V_\mu)$  is the action (3.2) minimally coupled to gravity. In the above expression  $R$  is the Ricci scalar and  $\mu^2$  is the cosmological constant. In two dimensions the curvature term in the gravitational action is purely topological, so at the classical level the metric  $g_{\mu\nu}$  acts as a Lagrange multiplier enforcing the constraint  $T_{\mu\nu} = 0$ . At the quantum level, in the conformal gauge  $g_{\mu\nu} = e^\phi \eta_{\mu\nu}$ , one obtains a conformal theory described by the Liouville action for the dilaton  $\phi$  coming from the measure in the functional integral, plus the action for matter (in the case at hand  $V_\mu, \lambda$ ) “gravitationally dressed” by the dilaton [39]–[43]. To see how the Einstein-aether sector contributes to the dilaton action let us have a closer look at the part of the functional integral involving  $V_\mu$  and  $\lambda$ . After fixing the conformal gauge it takes the form

$$Z_{EA} = \int \mathcal{D}_\phi \lambda \mathcal{D}_\phi V_\mu e^{i \int d^2x [\lambda(2V_+V_- - e^\phi) + \tilde{S}_{EA}(g_{\mu\nu}=e^\phi \eta_{\mu\nu}, V_\mu)]}. \quad (6.2)$$

where  $\tilde{S}_{EA}(g_{\mu\nu}, V_\mu)$  is the part of the action independent of the Lagrange multiplier  $\lambda$ . The  $\phi$ -subscript in the integration measure indicates that it is defined in such a way that (see, e.g., [44]),

$$\int \mathcal{D}_\phi \lambda e^{i \int d^2x \sqrt{|g|} \lambda^2} = \int \mathcal{D}_\phi \lambda e^{i \int d^2x e^\phi \lambda^2} = 1, \quad (6.3a)$$

$$\int \mathcal{D}_\phi V_\mu e^{i \int d^2x \sqrt{|g|} g^{\mu\nu} V_\mu V_\nu} = \int \mathcal{D}_\phi V_\mu e^{i \int d^2x \eta^{\mu\nu} V_\mu V_\nu} = 1. \quad (6.3b)$$

Equations (6.3) imply that the measure for the vector field  $V_\mu$  is  $\phi$ -independent. On the other hand, the measure for the Lagrange multiplier  $\lambda$  does depend on the dilaton  $\phi$ . This dependence is the same as for an ordinary two-dimensional scalar field. One gets rid of this dependence at the expense of introducing the Jacobian factor

$$e^{iJ(\phi)} = \det \left( e^{\phi(x)} \delta(x - x') \right),$$

which contributes to the Liouville action for the dilaton field [39]–[43]. Now it is straightforward to perform integration over the Lagrange multiplier  $\lambda$  that results in  $\delta(2V_+V_- - e^\phi)$ . Finally, one integrates over  $V_+$  and obtains

$$\begin{aligned} \int \mathcal{D}V_- | \det V_-^{-1} \delta(x - x') | e^{i\tilde{S}_{EA}(e^\phi \eta_{\mu\nu}, V_-, V_+ = e^\phi V_-^{-1})} \\ = \int \mathcal{D}\psi e^{i\tilde{S}_{EA}(e^\phi \eta_{\mu\nu}, V_- = e^\psi / \sqrt{2}, V_+ = e^{\phi-\psi} / \sqrt{2})}. \end{aligned} \quad (6.4)$$

Explicitly, the action in the last formula has the form

$$\tilde{S}_{EA}(\phi, \psi) = \int d^2x \frac{1}{g^2} \partial_+ \psi \partial_- (\psi - \phi) + \frac{\beta_{(+)}}{2g^2} e^{2\psi - \phi} (\partial_+ \psi)^2 + \frac{\beta_{(-)}}{2g^2} e^{\phi - 2\psi} (\partial_- (\psi - \phi))^2 . \quad (6.5)$$

To summarize, when coupled to gravity, in the conformal gauge the Einstein-aether sector contributes to the Liouville action in the same way as a single boson; additionally there is an interaction between the SO(1,1) Goldstone  $\psi$  and the dilaton due to the fact that the Einstein-aether action is not conformally invariant.

Note, that the dilaton does not receive a second order kinetic term with respect to the light cone time  $x^+$ . As discussed in section 2.2, this indicates a subtlety with the quantization of the theory (6.5) according to the  $x^+$ -ordered causal structure. Namely, the canonical formalism for this theory involves first class constraints related to the residual gauge invariance of the action (6.5). This gauge invariance manifests itself in the conformal symmetry, that at the classical level acts as

$$\phi(x^+, x^-) \mapsto \phi(f(x^+), g(x^-)) + \log f' + \log g' \quad (6.6)$$

$$\psi(x^+, x^-) \mapsto \psi(f(x^+), g(x^-)) + \log g' . \quad (6.7)$$

Clearly, more work is needed to understand the physics of (6.5). Note that the light cone gauge [45, 46] may turn out to be more appropriate than the conformal one for the analysis of theories with instantaneous causal structure coupled to gravity.

## 7. Conclusions and future directions

What are the main lessons to be drawn from our study? The parity breaking theories described here, to the best of our knowledge, present the first example of Lorentz invariant quantum field theories exhibiting non-local physics. Thus, they provide a theoretical laboratory for studying the effects of non-locality. Admittedly, our results crucially rely on the peculiar properties of a two dimensional world and there is no direct way to generalize them to higher dimensions. However, studies of two dimensional field theories proved to be useful for modeling dynamics in higher dimensions in many cases, such as confinement in QCD [47] and black hole information paradox [48]. Although at the level of intermediate calculations two dimensional physics is rather special in these cases as well, many of the final results turn out to be universal. So one may hope that also in the case of a superluminal travel two dimensional models may provide useful universally applicable lessons. In fact, the models considered in this paper share many of the properties of effective field theories describing spontaneous Lorentz breaking in higher dimensions [10, 11, 13].

Clearly, to proceed further in this direction the key question is whether the instantaneous theories remain consistent when coupled to gravity, and if the answer is yes, what the physical properties of the resulting theories are. It would be especially interesting to introduce the instantaneous causal structure in the dilaton gravity models possessing black hole solutions like those discussed in [48, 49]. If this can be done, one may obtain an example of a radical resolution of the black hole information paradox by allowing the information to escape already at the classical level, as it happens at the effective theory level in the Higgs phases of gravity in four dimensions [21].



There is another, even more intriguing, way to look at the instantaneous two dimensional models. Namely, in principle any model of two dimensional gravity can be thought of as a world-sheet theory for a (non-critical) string. In the present case one would obtain a string theory with instantaneous interactions propagating along the world-sheet. In such a situation one expects to find strong non-localities from the target space point of view as well, so this may provide a way to generate non-local gravitational theories in higher dimensions. In fact, if one thinks that effective field models describing gravity in the Higgs phase can be UV completed, this is a natural direction to proceed. Indeed, a natural way to obtain a weakly interacting quantum theory of a massless spin-2 particle is to start with a string. To find a deformation of such a theory one modifies the world-sheet dynamics. Usually, changing the world-sheet action corresponds just to taking a different background of the underlying string theory and such a modification is not radical enough to reproduce the physics of [10, 11, 13]. A drastic change of the causal structure of the world-sheet, if it can be implemented in a consistent manner, definitely appears as a more radical deformation of a theory. As an example of a possible world-sheet action one can take the action (3.10) with several flavors  $X^i$  ( $i = 1, \dots, D$ ) and couple it to gravity in the conformal gauge as in (6.5); this yields

$$S(\phi, \psi, X^i) = \int d^2x \left\{ \frac{1}{g^2} [\partial_+ \psi \partial_- (\psi - \phi) - \partial_+ X^i \partial_- X^i] + \frac{\beta}{2g^2} e^{2\psi - \phi} [(\partial_+ \psi)^2 + (\partial_+ X^i)^2] \right\} + S_L(\phi),$$

where  $S_L(\phi)$  is the Liouville action.

Both if one is interested in instantaneous theories coupled to 2d gravity as toy models for non-local gravitational theory, or as a world-sheet description of an exotic string theory, the crucial step is to construct conformal field theories exhibiting the instantaneous causal structure. At the classical level the action (7.1) is conformal, but it remains to be seen whether the conformal symmetry survives at the quantum level. Constructing a unitary conformal field theory with instantaneous causal structure would be interesting from yet another point of view. Indeed, using  $AdS_3/CFT_2$  duality such a theory should be dual to a three-dimensional gravitational theory with instantaneous signal propagation. This proposal is similar to that of [50], where it was also suggested to introduce instantaneous interaction on the CFT side as a way of generating non-local gravitational theories. The important difference is that in the current context this can, in principle, be achieved without breaking Poincare invariance.

To summarize, two-dimensional models with instantaneous Poincare invariant causal structure provide interesting examples of quantum field theories with non-local behavior. They provide a rich framework to study non-local physics, even in the presence of gravity. They suggest a number of intriguing opportunities, and we briefly outlined some of them. We feel that it is worth pursuing these ideas: irrespectively of whether they turn out to be successful or not, from this exercise one will learn more about both quantum field theory and gravity.

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## References

- [1] S.B. Giddings, *Quantization in black hole backgrounds*, *Phys. Rev. D* **76** (2007) 064027 [[hep-th/0703116](#)].
- [2] N. Arkani-Hamed, S. Dubovsky, A. Nicolis, E. Trincherini and G. Villadoro, *A measure of de Sitter entropy and eternal inflation*, *JHEP* **05** (2007) 055 [[arXiv:0704.1814](#)].
- [3] J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724 [[hep-th/9510017](#)].
- [4] S. Kachru, R. Kallosh, A. Linde and S.P. Trivedi, *De Sitter vacua in string theory*, *Phys. Rev. D* **68** (2003) 046005 [[hep-th/0301240](#)].
- [5] C. Vafa, *The string landscape and the swampland*, [hep-th/0509212](#).
- [6] N. Arkani-Hamed, L. Motl, A. Nicolis and C. Vafa, *The string landscape, black holes and gravity as the weakest force*, *JHEP* **06** (2007) 060 [[hep-th/0601001](#)].
- [7] H. Ooguri and C. Vafa, *On the geometry of the string landscape and the swampland*, *Nucl. Phys. B* **766** (2007) 21 [[hep-th/0605264](#)].
- [8] G. Dvali, M. Redi, S. Sibiryakov and A. Vainshtein, *Gravity cutoff in theories with large discrete symmetries*, *Phys. Rev. Lett.* **101** (2008) 151603 [[arXiv:0804.0769](#)].
- [9] G.R. Dvali, G. Gabadadze and M. Porrati, *4D gravity on a brane in 5D Minkowski space*, *Phys. Lett. B* **485** (2000) 208 [[hep-th/0005016](#)].
- [10] N. Arkani-Hamed, H.-C. Cheng, M.A. Luty and S. Mukohyama, *Ghost condensation and a consistent infrared modification of gravity*, *JHEP* **05** (2004) 074 [[hep-th/0312099](#)].
- [11] S.L. Dubovsky, *Phases of massive gravity*, *JHEP* **10** (2004) 076 [[hep-th/0409124](#)].
- [12] V.A. Rubakov and P.G. Tinyakov, *Infrared-modified gravities and massive gravitons*, [arXiv:0802.4379](#).
- [13] T. Jacobson and D. Mattingly, *Gravity with a dynamical preferred frame*, *Phys. Rev. D* **64** (2001) 024028 [[gr-qc/0007031](#)].
- [14] G. Dvali, A. Gruzinov and M. Zaldarriaga, *The accelerated universe and the Moon*, *Phys. Rev. D* **68** (2003) 024012 [[hep-ph/0212069](#)].
- [15] A. Lue and G. Starkman, *Gravitational leakage into extra dimensions: probing dark energy using local gravity*, *Phys. Rev. D* **67** (2003) 064002 [[astro-ph/0212083](#)].

- [16] S.L. Dubovsky, P.G. Tinyakov and I.I. Tkachev, *Massive graviton as a testable cold dark matter candidate*, *Phys. Rev. Lett.* **94** (2005) 181102 [[hep-th/0411158](#)].
- [17] P. Creminelli, M.A. Luty, A. Nicolis and L. Senatore, *Starting the universe: stable violation of the null energy condition and non-standard cosmologies*, *JHEP* **12** (2006) 080 [[hep-th/0606090](#)].
- [18] A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis and R. Rattazzi, *Causality, analyticity and an IR obstruction to UV completion*, *JHEP* **10** (2006) 014 [[hep-th/0602178](#)].
- [19] S.L. Dubovsky and S.M. Sibiryakov, *Spontaneous breaking of Lorentz invariance, black holes and perpetuum mobile of the 2nd kind*, *Phys. Lett.* **B 638** (2006) 509 [[hep-th/0603158](#)].
- [20] E. Babichev, V.F. Mukhanov and A. Vikman, *Escaping from the black hole?*, *JHEP* **09** (2006) 061 [[hep-th/0604075](#)].
- [21] S. Dubovsky, P. Tinyakov and M. Zaldarriaga, *Bumpy black holes from spontaneous Lorentz violation*, *JHEP* **11** (2007) 083 [[arXiv:0706.0288](#)].
- [22] C. Eling, B.Z. Foster, T. Jacobson and A.C. Wall, *Lorentz violation and perpetual motion*, *Phys. Rev.* **D 75** (2007) 101502 [[hep-th/0702124](#)].
- [23] E. Babichev, V. Mukhanov and A. Vikman, *k-essence, superluminal propagation, causality and emergent geometry*, *JHEP* **02** (2008) 101 [[arXiv:0708.0561](#)].
- [24] N.D. Mermin and H. Wagner, *Absence of ferromagnetism or antiferromagnetism in one-dimensional or two-dimensional isotropic Heisenberg models*, *Phys. Rev. Lett.* **17** (1966) 1133.
- [25] S.R. Coleman, *There are no Goldstone bosons in two-dimensions*, *Commun. Math. Phys.* **31** (1973) 259.
- [26] C. Eling and T. Jacobson, *Two-dimensional gravity with a dynamical aether*, *Phys. Rev.* **D 74** (2006) 084027 [[gr-qc/0608052](#)].
- [27] M. Visser, B. Bassett and S. Liberati, *Superluminal censorship*, *Nucl. Phys.* **88** (Proc. Suppl.) (2000) 267 [[gr-qc/9810026](#)].
- [28] S. Dubovsky, A. Nicolis, E. Trincherini and G. Villadoro, *Microcausality in curved space-time*, *Phys. Rev.* **D 77** (2008) 084016 [[arXiv:0709.1483](#)].
- [29] B.M. Gripaios, *Modified gravity via spontaneous symmetry breaking*, *JHEP* **10** (2004) 069 [[hep-th/0408127](#)].
- [30] M.V. Libanov and V.A. Rubakov, *More about spontaneous Lorentz-violation and infrared modification of gravity*, *JHEP* **08** (2005) 001 [[hep-th/0505231](#)].
- [31] D.S. Gorbunov and S.M. Sibiryakov, *Ultra-large distance modification of gravity from Lorentz symmetry breaking at the Planck scale*, *JHEP* **09** (2005) 082 [[hep-th/0506067](#)].
- [32] H.-C. Cheng, M.A. Luty, S. Mukohyama and J. Thaler, *Spontaneous Lorentz breaking at high energies*, *JHEP* **05** (2006) 076 [[hep-th/0603010](#)].
- [33] D.S. Gorbunov and S.M. Sibiryakov, *Self-accelerated brane universe with warped extra dimension*, [arXiv:0804.2248](#).
- [34] E. Witten, *Chiral symmetry, the  $1/n$  expansion and the  $SU(N)$  thirring model*, *Nucl. Phys.* **B 145** (1978) 110.

- [35] D. Nelson and R. Pelcovits, *Momentum-shell recursion relations, anisotropic spins, and liquid crystals*, *Phys. Rev.* **B 16** (1977) 2191.
- [36] A.M. Polyakov, *Interaction of Goldstone particles in two-dimensions. applications to ferromagnets and massive Yang-Mills fields*, *Phys. Lett.* **B 59** (1975) 79.
- [37] J. Polchinski, *Effective field theory and the Fermi surface*, [hep-th/9210046](#).
- [38] T.L. Curtright and C.B. Thorn, *Conformally invariant quantization of the Liouville theory*, *Phys. Rev. Lett.* **48** (1982) 1309 [*Erratum ibid.* **48** (1982) 1768].
- [39] A.M. Polyakov, *Quantum geometry of bosonic strings*, *Phys. Lett.* **B 103** (1981) 207.
- [40] F. David, *Conformal field theories coupled to 2D gravity in the conformal gauge*, *Mod. Phys. Lett.* **A 3** (1988) 1651.
- [41] J. Distler and H. Kawai, *Conformal field theory and 2D quantum gravity or who's afraid of Joseph Liouville?*, *Nucl. Phys.* **B 321** (1989) 509.
- [42] E. D'Hoker and P.S. Kurzepa, *2 – D quantum gravity and Liouville theory*, *Mod. Phys. Lett.* **A 5** (1990) 1411.
- [43] P.H. Ginsparg and G.W. Moore, *Lectures on 2 – D gravity and 2 – D string theory*, [hep-th/9304011](#).
- [44] O. Alvarez, *Theory of strings with boundaries: fluctuations, topology and quantum geometry*, *Nucl. Phys.* **B 216** (1983) 125.
- [45] A.M. Polyakov, *Quantum gravity in two-dimensions*, *Mod. Phys. Lett.* **A 2** (1987) 893.
- [46] V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, *Fractal structure of 2D-quantum gravity*, *Mod. Phys. Lett.* **A 3** (1988) 819.
- [47] G. 't Hooft, *A two-dimensional model for mesons*, *Nucl. Phys.* **B 75** (1974) 461.
- [48] A. Strominger, *Les Houches lectures on black holes*, [hep-th/9501071](#).
- [49] D. Grumiller, W. Kummer and D.V. Vassilevich, *Dilaton gravity in two dimensions*, *Phys. Rept.* **369** (2002) 327 [[hep-th/0204253](#)].
- [50] R. Sundrum, *Gravitational lorentz violation and superluminality via AdS/CFT duality*, *Phys. Rev.* **D 77** (2008) 086002 [[arXiv:0708.1871](#)].